An application of modified reductive perturbation method to long water waves

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Abstract: In this work, we extended the application of "the modified reductive perturbation method" to long water waves and obtained the conventional KdV equation for the lowest order term and the degenerate (linear) KdV equation with non-homogeneous part to the next term in the perturbation expansion. Seeking a localized travelling wave solutions to these evolution equations we determined the scale parameter c_1 so as to remove the possible secularities that might occur. The method can be extended to higher order expansion without any principal difficulty.

Keywords: *Modified reductive perturbation method, Water waves, Korteweg-de Vries equations.*

1. Introduction

In collisionless cold plasma, in fluid-filled elastic tubes and in shallow-water waves, due to nonlinearity of the governing equations, for weakly dispersive case one obtains the Korteweg-de Vries (KdV) equation for the lowest order term in the perturbation expansion, the solution of which may be described by solitons (Davidson [1]). To study the higher order terms in the perturbation expansion, the reductive perturbation method has been introduced by use of the stretched time and space variables (Taniuti [2]). However, in such an approach some secular terms appear which can be eliminated by introducing some slow scale variables (Sugimoto and Kakutani [3]) or by a renormalization procedure of the velocity of the KdV soliton (Kodama and Taniuti [4]). Nevertheless, this approach remains somewhat artificial, since there is no reasonable connection between the smallness parameters of the stretched variables and the one used in the perturbation expansion of the field variables. The choice of the former parameter is based on the linear wave analysis of the concerned problem and the wave number or the frequency is taken as the perturbation parameter (Washimi and Taniuti [5]). On the other hand, at the lowest order, the amplitude and the width of the wave are expressed in terms of the unknown perturbed velocity, which is also used as the smallness parameter. This causes some ambiguity over the correction terms. Another attempt to remove such secularities is made by Kraenkel et al [6] for long water waves by use of the multiple time scale expansion but could not obtain explicitly the correction terms to the wave speed.

In order to remove these uncertainities, Malfliet and Wieers [7] presented a dressed solitary wave approach, which is based on the assumption that the field variables admit localized travelling wave solution. Then, for the longwave limit, they expanded the field variables and the wave speed into a power series of the wave number, which is assumed to be the only smallness parameter, and obtained the explicit solution for various order terms in the expansion. However, this approach can only be used when one studies progressive wave solution to the original nonlinear equations and it does not give any idea about the form of evolution equations governing the various order terms in the perturbation expansion. In our previous paper [8], we have presented a method so called "the modified reductive perturbation method" to examine the contributions of higher order terms in the perturbation expansion and applied it to weakly dispersive ion-acoustic plasma waves.

In this work, we extended the application of "the modified reductive perturbation method" developed by us [8]to long water waves and obtained the conventional KdV equation for the lowest order term in perturbation expansion and the degenerate (linear) KdV equation with non-homogeneous part to the next term in the perturbation expansion. Seeking a localized travelling wave solutions to these evolution equations we determined the scale parameter c_1 so as to remove the possible secularities that might occur. The present method is seen to be fairly simple as compared to the renormalization method of Kodama and Taniuti [4] and the multiple scale expansion method of Kraenkel et al [6]. The method can be extended to higher order expansion without any principal difficulty.

2. Modified reductive perturbation formalism for water waves

We consider a two-dimensional incompressible inviscid fluid of height h_0 in a constant gravitational field g acting in negative z direction. The space coordinates are denoted by (x, z) and the corresponding velocity components by (u, w). Following Demiray [9], the equations describing potential flow are given by

$$\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial z^2} = 0, \tag{1}$$

with the boundary conditions

$$\frac{\partial \hat{\phi}}{\partial z} = \frac{\partial \hat{\psi}}{\partial t} + \frac{\partial \hat{\phi}}{\partial x}\frac{\partial \hat{\psi}}{\partial x}, \quad \text{on} \quad z = \hat{\psi}(x, t). \tag{2}$$

$$\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2}[(\frac{\partial \hat{\phi}}{\partial x})^2 + (\frac{\partial \hat{\phi}}{\partial z})^2] + g\hat{\psi} = 0, \quad \text{on} \quad z = \hat{\psi}(x, t). \tag{3}$$

$$\frac{\partial \hat{\phi}}{\partial z} = 0 \quad \text{at} \quad z = -h_0. \tag{4}$$

Here $\hat{\phi}(x, z, t)$ characterizes the flow potential and $\hat{\psi}(x, t)$ stands for the free surface function of the fluid. Of these boundary conditions, the equations (2) and (3), respectively, describe the kinematical and the dynamical boundary conditions on the free surface $z = \hat{\psi}(x, t)$ and the equation (4) states that the normal velocity should be equal to zero on the rigid bottom $z = -h_0$.

Now, we shall consider the long wave in shallow-water approximation to the above equations by applying the modified reductive perturbation method developed by us (Demiray [8]). According to this method, we introduce the following coordinate stretching

$$\xi = \epsilon^{1/2}(x - c_0 t), \quad \tau = \epsilon^{3/2} c x, \tag{10}$$

where ϵ is a small parameter characterizing the smallness of certain physical entities, c_0 and c are two constants to be determined from the solution.

We expand the functions $\hat{\phi}$ and $\hat{\psi}$, and the constant c into a suitable power series in the parameter ϵ :

$$\hat{\phi} = \epsilon^{1/2}(\phi_0+\epsilon\phi_1+\epsilon^2\phi_2+\epsilon^3\phi_3+.), \quad \hat{\psi} = \epsilon(\psi_0+\epsilon\psi_1+\epsilon^2\psi_2+\epsilon^3\psi_3+.), \quad c = 1+\epsilon c_1+.... \tag{11}$$

Introducing the expansions (11) into equations (1)-(4)and setting the coefficients of alike powers of ϵ equal to zero, the following set of differential equations are obtained:

$O\ (1)$ *equations*

$$\frac{\partial^2\phi_0}{\partial z^2} = 0, \tag{12}$$

and the boundary conditions

$$\frac{\partial\phi_0}{\partial z}|_{z=-h_0} = 0, \quad \frac{\partial\phi_0}{\partial z}|_{z=0} = 0, \quad \frac{\partial\phi_0}{\partial\xi}|_{z=0} + g\psi_0 + \frac{1}{2}(\frac{\partial\phi_0}{\partial z})^2|_{z=0} = 0. \tag{13}$$

$O(\epsilon)$ *equations*

$$\frac{\partial^2\phi_1}{\partial z^2} + c_0^2\frac{\partial^2\phi_0}{\partial\xi^2} = 0, \tag{14}$$

and the boundary conditions

$$\frac{\partial\phi_1}{\partial z}|_{z=-h_0} = 0, \quad \frac{\partial\phi_1}{\partial z}|_{z=0} - \frac{\partial\psi_0}{\partial\xi} = 0, \quad [\frac{\partial\phi_1}{\partial\xi} + \frac{\partial\phi_0}{\partial z}\frac{\partial\phi_1}{\partial z} + \frac{c_0^2}{2}(\frac{\partial\phi_0}{\partial\xi})^2]|_{z=0} + g\psi_1 = 0. \tag{15}$$

$O(\epsilon^2)$ *equations*

$$\frac{\partial^2\phi_2}{\partial z^2} + c_0^2\frac{\partial^2\phi_1}{\partial\xi^2} - 2c_0\frac{\partial^2\phi_0}{\partial\xi\partial\tau} = 0, \tag{16}$$

and the boundary conditions

$$\frac{\partial\phi_2}{\partial z}|_{z=-h_0} = 0, \quad [\frac{\partial\phi_2}{\partial z} + \psi_0\frac{\partial^2\phi_1}{\partial z^2}]|_{z=0} - \frac{\partial\psi_1}{\partial\xi} - c_0^2\frac{\partial\phi_0}{\partial\xi}|_{z=0}\frac{\partial\psi_0}{\partial\xi} = 0$$

$$[\frac{\partial\phi_2}{\partial\xi} + \psi_0\frac{\partial^2\phi_1}{\partial z\partial\xi} + \frac{1}{2}(\frac{\partial\phi_1}{\partial z})^2 + \frac{\partial\phi_0}{\partial z}\frac{\partial\phi_2}{\partial z} - c_0\frac{\partial\phi_0}{\partial\xi}(-c_0\frac{\partial\phi_1}{\partial\xi} + \frac{\partial\phi_0}{\partial\tau})]|_{z=0} + g\psi_2 = 0. \tag{17}$$

$O(\epsilon^3)$ *equations*

$$\frac{\partial^2\phi_3}{\partial z^2} + c_0^2\frac{\partial^2\phi_2}{\partial\xi^2} - 2c_0\frac{\partial^2\phi_1}{\partial\xi\partial\tau} - 2c_0c_1\frac{\partial^2\phi_0}{\partial\xi\partial\tau} + \frac{\partial^2\phi_0}{\partial\tau^2} = 0, \tag{18}$$

and the boundary conditions

$$\frac{\partial\phi_3}{\partial z}|_{z=-h_0} = 0, \quad [\frac{\partial\phi_3}{\partial z} + \psi_0\frac{\partial^2\phi_2}{\partial z^2} + \psi_1\frac{\partial^2\phi_1}{\partial z^2}]|_{z=0} - \frac{\partial\psi_2}{\partial\xi}$$

$$+c_0(-c_0\frac{\partial\psi_1}{\partial\xi}+\frac{\partial\psi_0}{\partial\tau})\frac{\partial\phi_0}{\partial\xi}|_{z=0}+c_0\frac{\partial\psi_0}{\partial\xi}(-c_0\frac{\partial\phi_1}{\partial\xi}+\frac{\partial\phi_0}{\partial\tau})|_{z=0}=0, \tag{19}$$

2.1. Solution of the field equations

From the solution of the sets (12) and (13) we have

$$\phi_0=\varphi(\xi,\tau),\quad \psi_0=-\frac{1}{g}\frac{\partial\varphi}{\partial\xi}, \tag{20}$$

where $\varphi(\xi,\tau)$ is an unknown function of its argument whose governing equation will be obtained later.

Similarly, from the solution of the equations (14) and (15) one obtains

$$\phi_1=-\frac{c_0^2}{2}\frac{\partial^2\varphi}{\partial\xi^2}(z^2+2h_0z)+\varphi_1(\xi,\tau),\quad \psi_1=-\frac{1}{g}\frac{\partial\varphi_1}{\partial\xi}-\frac{c_0^2}{g}(\frac{\partial\varphi}{\partial\xi})^2,\quad c_0=(gh_0)^{-1/2}, \tag{21}$$

where $\varphi_1(\xi,\tau)$ is another unknown function whose governing equation will be obtained from the higher order expansions.

The solution of $O(\epsilon^2)$ equations, (16) and (17), yields the following results

$$\phi_2=\frac{c_0^4}{24}\frac{\partial^4\varphi}{\partial\xi^4}(z^4+4hz^3)+(c_0\frac{\partial^2\varphi}{\partial\xi\partial\tau}-\frac{c_0^2}{2}\frac{\partial^2\varphi_1}{\partial\xi^2})z^2-(3\frac{c_0^2}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi^2}+\frac{1}{g}\frac{\partial^2\varphi_1}{\partial\xi^2})z+\varphi_2,$$

$$\psi_2=-\frac{1}{g}\frac{\partial\varphi_2}{\partial\xi}-\frac{c_0^2h_0}{g^2}\frac{\partial\varphi}{\partial\xi}\frac{\partial^3\varphi}{\partial\xi^3}-\frac{c_0^4h_0^2}{2g}(\frac{\partial^2\varphi}{\partial\xi^2})^2+\frac{c_0}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi}{\partial\tau}-\frac{c_0^2}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi_1}{\partial\xi}, \tag{22}$$

where $\varphi_2(\xi,\tau)$ is another unknown function whose governing equation will be obtained later. The use of the last boundary condition and setting $\partial\varphi/\partial\xi=-g\psi_0$, yields the following Korteweg-deVries equation

$$\frac{\partial\psi_0}{\partial\tau}-\frac{3c_0^3g}{2}\psi_0\frac{\partial\psi_0}{\partial\xi}-\frac{c_0^3h_0^2}{6}\frac{\partial^3\psi_0}{\partial\xi^3}=0. \tag{23}$$

In order to see the novelty of the present method we should study the $O(\epsilon^3)$ equations. In order to save the space the detail of the solution will not be given here. From the solution of equations (18) and (19), the following evolution equation can be obtained for the second order term in the perturbation expansion

$$\frac{\partial\psi_1}{\partial\tau}-\frac{3c_0^3g}{2}\frac{\partial}{\partial\xi}(\psi_0\psi_1)-\frac{h_0^2c_0^3}{6}\frac{\partial^3\psi_1}{\partial\xi^3}=S(\varphi), \tag{24}$$

where $S(\varphi)$ is defined by

$$S(\varphi)=\frac{c_0^5h_0^4}{60g}\frac{\partial^6\varphi}{\partial\xi^6}+\frac{c_0^2h_0^2}{6g}\frac{\partial^4\varphi}{\partial\xi^3\partial\tau}-\frac{3c_0^5h_0^2}{4g}\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial^3\varphi}{\partial\xi^3}-\frac{c_0^5h_0^2}{12g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^4\varphi}{\partial\xi^4}$$

$$+\frac{c_1}{g}\frac{\partial^2\varphi}{\partial\xi\partial\tau}-\frac{3c_0^2}{g}\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi\partial\tau}-\frac{c_0^2}{g}\frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial\varphi}{\partial\tau}+\frac{3c_0^5}{g}(\frac{\partial\varphi}{\partial\xi})^2\frac{\partial^2\varphi}{\partial\xi^2}-\frac{1}{2c_0g}\frac{\partial^2\varphi}{\partial\tau^2}. \tag{25}$$

The equation (24) is the degenerate(linearized) KdV equation with non-homogeneous term $S(\varphi)$. It is seen that this non-homogeneous term contains the unknown coefficient c_1, which is to be determined from the removal of some possible secularities that might occur. This will be investigated in the following sub-section.

2.2. Progressive wave solution

In this section we shall present a localized travelling wave solutions to equations (23) and (24). For that purpose we shall seek a solution to these equations in the following form

$$\psi_0 = U(\zeta), \quad \psi_1 = V(\zeta), \quad \zeta = \alpha(\xi + \beta\tau), \tag{26}$$

where α and β are two constants to be determined from the solutions. As is well known the conventional KdV equation assumes the solution of the form

$$U = a\ \mathrm{sech}^2\zeta, \tag{27}$$

where a is the amplitude of the solitary wave and the other quantities are found to be

$$\alpha = \frac{1}{2h_0}(3ag)^{1/2}, \quad \beta = \frac{c_0 a}{2h_0}. \tag{28}$$

Here, we note that the wave speed is proportional to the amplitude of the wave. Introducing the solutions $(26)_2$ and (27) into the evolution equations(24) and (25) we have

$$V'' + (12\mathrm{sech}^2\zeta - 4)V = (\frac{11}{5}\frac{a^2}{h_0} + 4ac_1)\mathrm{sech}^2\zeta + \frac{6a^2}{h_0}\mathrm{sech}^4\zeta + \frac{6a^2}{h_0}\mathrm{sech}^6\zeta. \tag{29}$$

Here, we shall propose a solution for V of the following form

$$V = A\ \mathrm{sech}^4\zeta + B\ \mathrm{sech}^2\zeta, \tag{30}$$

where A and B are two constants to be determined from the solution. As is seen from equations (29) and (30) there will not be any term in V to balance $\mathrm{sech}^2\zeta$ on the right hand side of (29). Thus, in order to remove this secularity the coefficient of $\mathrm{sech}^2\zeta$ must vanish, which yields

$$c_1 = -\frac{11}{20}\frac{a}{h_0}. \tag{31}$$

Here c_1 is the first order correction term to the wave speed. Substitution of (30) into (29) gives

$$A = -\frac{3a^2}{4h_0}, \quad B = \frac{5a^2}{2h_0}, \tag{32}$$

Thus, the total solution takes the following form

$$\psi = \psi_0 + \epsilon\psi_1 = a\mathrm{sech}^2\zeta + \epsilon(-\frac{3a^2}{4h_0}\mathrm{sech}^4\zeta + \frac{5a^2}{2h_0}\mathrm{sech}^2\zeta), \tag{33}$$

with

$$\zeta = \frac{(3ag)^{1/2}}{2h_0}\{\epsilon^{1/2}[t - c_0(1 - \frac{a}{2h_0}\epsilon + \frac{11}{40}\frac{a^2}{h_0^2}\epsilon^2)x]\}. \tag{34}$$

The propagation speed can, now, be defined by

$$v_p = \frac{dx}{dt} = \frac{1}{c_0}\frac{1}{(1 - \frac{a}{2h_0}\epsilon + \frac{11a^2}{40h_0^2}\epsilon^2)} \approx (gh_0)^{1/2}(1 + \frac{a}{2h_0}\epsilon - \frac{a^2}{40h_0^2}\epsilon^2 + ...). \tag{35}$$

Concluding Remarks

The study of the effects of higher order terms in the perturbation expansion of field quantities through the use of classical reductive perturbation method leads to some secularities. To eliminate such secularities various methods, like renormalization method of Kodama and Taniuti [4], the multiple scale expansion method by Kraenkel et al[6], have been presented in the current literature. The results of present work and of those given in reference [8] proved that the "modified reductive perturbation method" presented by us is the most simplest and the effective one. By use of this method, any order of correction term may be obtained without any principal difficulties.

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